Resit Graph Theory

12 March 2026, 18:30–20:30

- It is not allowed to use calculators, phones, computers, books, notes, the help of others or any other aids.
- Make sure to state clearly any results from the lecture notes you are using.
- Write the answer to each question on a separate sheet, **with your name and student number on each sheet.** This is worth 10 points (out of a total of 100). Nota bene: by a 'sheet' we mean a folded booklet with the university logo on the front.

Exercise 1 (20 points)
Consider a scenario with 4 professors and 4 students. Each professor will supervise the project of one student. Each professor has ranked the students and each student has ranked the professors.

1. Give the definition of a stable matching in this scenario.
2. Determine the result of the Gale–Shapley algorithm (in which the students propose to the professors) on the situation below. Make sure to clearly indicate, for each step of the algorithm, what actions are taken by the algorithm.

	professors' preferences
p_1 :	$s_3 > s_4 > s_1 > s_2$
p_2 :	$s_4 > s_2 > s_3 > s_1$
p_3 :	$s_1 > s_2 > s_4 > s_3$
p_4 :	$s_4 > s_3 > s_1 > s_2$

	students' preferences
s_1 :	$p_1 > p_3 > p_2 > p_4$
s_2 :	$p_3 > p_4 > p_1 > p_2$
s_3 :	$p_1 > p_3 > p_2 > p_4$
s_4 :	$p_3 > p_2 > p_1 > p_4$

Exercise 2 (25 points)
Let G be a graph and let $\delta(G)$ be the minimum degree in G. Let k be a natural number with $k \geq 2$.

1. Show that if G is k-connected then $\delta(G) \geq k$.
2. Give an example of a graph with $\delta(G) \geq k$ that is not k-connected.
3. Show that if $\delta(G) \geq (v(G) + k)/2$ then G is k-connected.

Exercise 3 (25 points)

1. Let G be a Hamiltonian bipartite graph, and choose $x, y \in V(G)$. Prove that $G \setminus \{x, y\}$ has a perfect matching if and only if x and y are in different parts of the bipartition.
2. Prove that deleting two unit squares from an $2n \times 2n$ chessboard leaves a board that can be partitioned into 1×2 rectangles if and only if the two missing squares have different colours.

Exercise 4 (20 points)
Let $n \geq 2$. Consider n lines in the plane so that each pair of lines intersect, and no three lines intersect at the same point. Let G be the graph whose vertices are the intersections of the lines, with two vertices adjacent if they appear consecutively on one of the lines.

1. Show that G is planar and show that $f(G) = 2 + n(n-2) - n(n-1)/2$.
2. Without using Thomassen's theorem on the list chromatic number of planar graphs, prove that $\chi_\ell(G) \leq 5$.

   Bonus (+10 points): Show that $\chi_\ell(G) \leq 3$.

(The end)